Journal of Statistical Physics, Vol. 45, Nos. 3/4, 1986

Some Critical Exponent Inequalities for Percolation

C. M. Newman[1]

Received May 13, 1986; revision received July 29, 1986

For a large class of independent (site or bond, short- or long-range) percolation models, we show the following: (1) If the percolation density $P_\infty(p)$ is discontinuous at p_c, then the critical exponent γ (defined by the divergence of expected cluster size, $\sum nP_n(p) \sim (p_c - p)^{-\gamma}$ as $p \uparrow p_c$) must satisfy $\gamma \geqslant 2$. (2) γ or γ' (defined analogously to γ, but as $p \downarrow p_c$) and δ $[P_n(p_c) \sim n^{-1-1/\delta}$ as $n \to \infty]$ must satisfy $\gamma, \gamma' \geqslant 2(1 - 1/\delta)$. These inequalities for γ improve the previously known bound $\gamma \geqslant 1$ (Aizenman and Newman), since $\delta \geqslant 2$ (Aizenman and Barsky). Additionally, result 1 *may* be useful, in standard d-dimensional percolation, for proving rigorously (in $d > 2$) that, as expected, P_∞ has no discontinuity at p_c.

KEY WORDS: Percolation; critical exponent inequalities; rigorous results.

1. INTRODUCTION AND RESULTS

1.1. Background

We are concerned in this paper with two interconnected kinds of results for percolation. The first involves the relation between the divergence as $p \uparrow p_c$ of the expected cluster size $\chi(p)$ [as described by the exponent γ: $\chi(p) \sim (p_c - p)^{-\gamma}$] and the vanishing of $P_\infty(p_c)$, the percolation density at the critical point. The second involves the relation between γ or other similar exponents and the decay as $n \to \infty$ of $P_n(p_c)$, the cluster size distribution at the critical point [as described by the exponent δ: $P_n(p_c) \sim n^{-1-1/\delta}$].

Our results of the first kind may be regarded from two perspectives. Both perspectives implicitly make use of the fact, proved recently by Aizenman and Barsky[1] to be generally valid, that there is a *single* critical point

[1] Department of Mathematics, University of Arizona, Tucson, Arizona 85721.

0022-4715/86/1100-0359$05.00/0

p_c above which P_∞ is positive and below which χ is finite. The first perspective is from the context of one-dimensional $1/|x-y|^2$ models, where it has been proved that there is a phase transition (with $p_c \neq 1$)[17] and that $P_\infty(p_c) > 0$[6]: for such models we show that $\gamma \geqslant 2$. The second perspective is from the context of most other models, such as standard site or bond percolation in dimension $d > 2$, where it is expected, but not yet rigorously proved, that $P_\infty(p_c) = 0$: for such models we show that to derive continuity of $P_\infty(p)$ at p_c, it suffices to show that $\gamma < 2$. Note that for $d = 3$, γ is numerically estimated to be about 1.7 (see, e.g., Ref. 19 and the references given there). We also remark that P_∞ has been proven to be continuous for all $p > p_c$.[4,7]

Our main result of the second kind is that γ and the analogous (as $p \downarrow p_c$) exponent γ' satisfy $\gamma, \gamma' \geqslant 2(1 - 1/\delta)$. Since $\delta \geqslant 2$,[1] this improves the previous result[5] that $\gamma \geqslant 1$. Other rigorous inequalities are known, which involve the exponent β $[P_\infty(p) \sim (p-p_c)^\beta$ as $p \downarrow p_c]$: $\beta \leqslant 1$[8] and $\beta(\delta - 1) \geqslant 1$. The latter inequality was derived for Ising models in Ref. 13 as a consequence of a differential inequality related to Burgers' equation; the differential inequality and hence the exponent inequality were extended to percolation models in Ref. 1. Some inequalities have also been obtained for the "specific heat" exponent α.[4] For standard two-dimensional models, there exist nonrigorous but presumably exact values for the critical exponents[15,16,18] as well as rigorous hyperscaling identities among exponents.[11,12]

In the remainder of this section, we describe the class of percolation models we consider and then precisely state our results. In Section 2, we give the essential ingredients of the proofs. For more details, see Ref. 14.

1.2. Setup

All our results concern independent translation-invariant site or bond percolation. For simplicity, the site models we consider will be standard nearest neighbor percolation on the hypercubic lattice $\mathbb{Z}^d$ with site occupation probability p. We denote by $\mathbb{N} \geqslant 0$ the size of the cluster of the origin, i.e., the number of occupied sites connected to the origin by nearest neighbor paths touching only occupied sites.

The bond models we consider have bonds $\{x, y\}$ between pairs of sites in $\mathbb{Z}^d$ which are independently occupied with probability p_{x-y} (not equal to one). Here $\mathbb{N} \geqslant 1$ denotes the number of *sites* connected to the origin by paths of occupied bonds. We choose some finite collection of sites (invariant under $z \to -z$) and set $p_z = p$ for all z in that collection; for every other z, p_z is held fixed as p is varied. The standard nearest neighbor bond model, for example, takes the nearest neighbors of the origin as its collection and sets $p_z = 0$ for every other z. In long-range, one-dimensional models, one often assumes $p_{x-y} \sim |x-y|^{-s}$ for some $s > 0$ as $|x-y| \to \infty$, and chooses $p_1 = p_2 = \cdots = p_R = p$ for some convenient R.[6,17]

For both site and bond percolation, we make the following standard definitions of the percolation density P_∞, cluster size distribution P_n, critical point p_c, and expected cluster size χ (or χ'):

$$P_n(p) = \text{Prob}_p(\mathbb{N} = n) \qquad \text{for} \quad n \leqslant \infty \tag{1.1}$$

$$p_c = \sup\{p: P_\infty(p) = 0\} \tag{1.2}$$

$$\chi(p) = E_p(\mathbb{N}) = \sum_{n \leqslant \infty} nP_n(p) \tag{1.3}$$

$$\chi'(p) = \sum_{n < \infty} nP_n(p) \tag{1.4}$$

We will always assume that $0 < p_c < 1$. This requires $\sum p_z < \infty$, and either that $d > 1$ or else, for $d = 1$ bond percolation, essentially that $\lim z^2 p_z > 1$.[6,17]

It has recently been proven[1] that in all these models $\chi(p) < \infty$ for any $p < p_c$. Of course, $\chi' = \chi$ for $p < p_c$, but for $p > p_c$, $\chi = \infty$, while χ' is believed to be finite for most models. However, no general theorem guarantees this belief; indeed, it is known not to be so in some $1/|x-y|^2$ models.[3]

1.3. Results

Our first theorem relates the divergence of $\chi(p)$ as $p \uparrow p_c$ to the vanishing or nonvanishing of $P_\infty(p_c)$, i.e., to whether $P_\infty(p)$ has a discontinuous transition at p_c. As noted above, such a discontinuity does occur in $1/|x-y|^2$ models.[6] The theorem is not stated directly in terms of γ so as to avoid any assumptions as to whether, and in what sense, $\chi(p) \sim (p_c - p)^{-\gamma}$ as $p \uparrow p_c$. However, any reasonable version of such an assumption, combined with the theorem, would yield

$$P_\infty(p_c) > 0 \qquad \text{implies} \quad \gamma \geqslant 2 \tag{1.5}$$

or equivalently

$$\gamma < 2 \qquad \text{implies} \quad P_\infty(p_c) = 0 \tag{1.6}$$

The inequality $\gamma \geqslant 2$ of (1.5) makes its appearance in the statement of the theorem in the guise of a divergence criterion [Eq. (1.7)]. If one wishes, that criterion can be replaced (at the cost of slightly weakening the theorem) by a less disguised version of $\gamma \geqslant 2$, namely,

$$\text{for any} \quad \varepsilon > 0, \qquad \limsup_{p \uparrow p_c} (p_c - p)^{2-\varepsilon} \chi(p) = \infty$$

Theorem 1. If $0 < p_c < 1$ and $P_\infty(p_c) > 0$, then

$$\int_0^{p_c - 0} [\chi(p)]^{1/2}\, dp = \infty \tag{1.7}$$

The proof of Theorem 1, which will be presented in Section 2, is quite comprehensible and the reader is urged to read it.

Remarks. (i) In bond models, one often considers a different choice of parameter than the p defined above. Namely, one may set

$$p_z = 1 - \exp(-\beta J_z)$$

with free parameter β (not to be confused with the critical exponent β) and with all J_z held fixed as β is varied. If $0 < \beta_c < \infty$ (which requires $\sum J_z < \infty$), it can be shown that

$$P_\infty(\beta_c) > 0 \qquad \text{implies} \quad \int_0^{\beta_c - 0} [\chi(\beta)]^{1/2}\, d\beta = \infty$$

When there are infinitely many nonzero J_z, this does not seem to follow from Theorem 1, but it can be proven by similar arguments (see Ref. 14 for details).

(ii) Theorem 1 can be combined with a result from Ref. 5 to give an alternate proof of the recently derived fact[(2)] that $P_\infty(p_c) = 0$ whenever the "triangle criterion" is satisfied. The triangle criterion. introduced in Ref. 5 and expected to be valid in short-range models above six dimensions, states that the two-point connectivity function, defined as

$$\tau(x, y) = \text{Prob}_p\,(x \text{ and } y \text{ belong to the same cluster})$$

satisfies

$$\sum_x \sum_y \tau(0, x)\, \tau(x, y)\, \tau(y, 0) < \infty \qquad \text{at} \quad p = p_c$$

It was shown in Ref. 5 that when the triangle criterion is satisfied, $\gamma = 1$ [in the sense that χ is bounded above (and below) by a constant times $(p_c - p)^{-1}$ as $p \uparrow p_c$] and hence, by Theorem 1, $P_\infty(p_c) = 0$, since the integral in (1.7) is convergent. Another alternate proof, based on the uniqueness of infinite clusters, may be found in Ref. 4; that argument yields sufficiency conditions for $P_\infty(p_c) = 0$ that are much weaker than the triangle criterion.

Our second theorem assumes a one-sided version of $P_n(p_c) \sim n^{-1-1/\delta}$ and concludes with a precise version of the inequalities

$$\gamma \geqslant 2(1-1/\delta), \qquad \gamma' \geqslant 2(1-1/\delta) \tag{1.8}$$

The proof will be given in Section 2.

Theorem 2. If $0 < p_c < 1$ and if for some $\delta > 1$ and $B_1 > 0$,

$$P_n(p_c) \geqslant B_1 n^{-(1+1/\delta)} \qquad \text{as} \quad n \to \infty \tag{1.9}$$

then for some $B_2 > 0$

$$\chi(p_c - \varepsilon), \chi'(p_c + \varepsilon) \geqslant B_2 |\varepsilon^2 \log(|\varepsilon|)|^{-(1-1/\delta)} \qquad \text{as} \quad \varepsilon \downarrow 0 \tag{1.10}$$

Remarks. (i) Let us define for $r > 0$,

$$\chi_r(p) = E_p(\mathbb{N}^r) = \sum_{n \leqslant \infty} n^r P_n(p) \tag{1.11}$$

and $\chi'_r(p)$ analogously. The proof of Theorem 2 automatically yields inequalities on the corresponding critical exponents:

$$\gamma_r, \gamma'_r \geqslant 2(r - 1/\delta) \qquad \text{for} \quad r > 1/\delta \tag{1.12}$$

Now $\chi_r(p)$ and $\chi'_r(p)$ are (by Hölder's inequality) log-convex functions of r, so that γ_r and γ'_r (assuming they exist in some reasonable sense) will be convex in r. Moreover, if δ exists in a reasonable sense, then one should have $\gamma_{1/\delta} = 0 = \gamma'_{1/\delta}$. Convexity would then imply an improvement of (1.12),

$$\gamma_r, \gamma'_r \geqslant \frac{\gamma}{1 - 1/\delta}\left(r - \frac{1}{\delta}\right) \qquad \text{for} \quad r > 1 \tag{1.13}$$

and would also imply

$$\gamma_r, \gamma'_r \leqslant \frac{\gamma}{1 - 1/\delta}\left(r - \frac{1}{\delta}\right) \qquad \text{for} \quad \frac{1}{\delta} < r < 1 \tag{1.14}$$

(ii) The logarithmic factor in (1.10) can be eliminated at the cost of mixing together γ and γ'; e.g., (1.10) can be replaced by

$$[\chi(p_c - \varepsilon)\, \chi'(p_c + \varepsilon + O(\varepsilon^2))]^{1/2} \geqslant B_3 |\varepsilon|^{-2(1-1/\delta)} \qquad \text{as} \qquad \varepsilon \downarrow 0 \tag{1.15}$$

This inequality is valid even if the hypothesis (1.9) on $P_n(p_c)$ is weakened to

$$\sum_{n < \infty} e^{-nh} n P_n(p_c) \geqslant B'_1 h^{1/\delta - 1} \qquad \text{as} \quad h \downarrow 0 \tag{1.16}$$

See the remarks at the end of Ref. 14, Section 3 for more details.

2. DERIVATIONS

We will present here the derivations only for the case of site percolation. The proofs for bond percolation are essentially the same, but with some extra complications. See Ref. 14 for more details.

Proof of Theorem 1 (for site percolation). We use the standard identities

$$P_\infty(p) = 1 - \sum_{n<\infty} P_n(p) \tag{2.1}$$

and

$$P_n(p) = \sum_l a_{nl} p^n (1-p)^l \equiv \sum_l P_{nl}(p) \tag{2.2}$$

where a_{nl} is the number of lattice animals with n occupied sites and l vacant boundary sites. These imply that

$$(d/dp)\, P_n(p) = \sum_l \left[n/p - l/(1-p) \right] P_{nl}(p) \tag{2.3}$$

and that

$$\begin{aligned} P_\infty(p_c) &= -\lim_{N\to\infty} \sum_{n<N} \left[P_n(p_c) - P_n(p_c - \varepsilon) \right] \\ &= \lim_{N\to\infty} \int_{p_c-\varepsilon}^{p_c-0} \left[\sum_{n<N} (-d/dp)\, P_n(p) \right] dp \\ &= \lim_{N\to\infty} \int_{p_c-\varepsilon}^{p_c-0} \left\{ \sum_{\substack{l\\ n<N}} -\left[n/p - l/(1-p) \right] P_{nl}(p) \right\} dp \\ &\leqslant \lim_{N\to\infty} \int_{p_c-\varepsilon}^{p_c-0} \left\{ \sum_{\substack{l\\ n<N}} \left[n/p - l/(1-p) \right]^2 P_{nl}(p) \right\}^{1/2} dp \end{aligned} \tag{2.4}$$

where the last step uses the Cauchy–Schwarz inequality (for sequences indexed by n and l) and the intermediate steps use the fact that $\sum_{n<N} P_n(p)$ is a smooth function of p (since for each n, only finitely many a_{nl} are nonzero) even though $P_\infty(p)$ is not smooth (at p_c).

Next we differentiate (2.1) for $p < p_c$ once to obtain

$$\sum_{\substack{l\\ n<\infty}} \left[n/p - l(1-p) \right] P_{nl}(p) = 0 \tag{2.5}$$

and twice to obtain

$$\sum_{\substack{l \\ n<\infty}} [n/p - l/(1-p)]^2 P_{nl}(p) = \sum_{\substack{l \\ n<\infty}} [n/p^2 + l/(1-p)^2] P_{nl}(p)$$

$$= p^{-2}(1-p)^{-1} \chi(p) \qquad (2.6)$$

where the last equality uses (2.5). There are no convergence problems in (2.5)–(2.6) because for $p < p_c$, P_n decays exponentially fast in n.[9] Combining (2.6) with (2.4) yields

$$P_\infty(p_c) \leqslant \int_{p_c-\varepsilon}^{p_c-0} [p^{-2}(1-p)^{-1} \chi(p)]^{1/2}\, dp \qquad (2.7)$$

Letting $\varepsilon \to 0$ shows that if $[\chi(p)]^{1/2}$ has a finite integral over $(0, p_c)$, then $P_\infty(p_c) = 0$, which completes the proof. ∎

Before giving the precise proof of Theorem 2 (for site percolation), we sketch the basic ideas behind it. These have about them the general flavor of standard scaling theory (see, e.g., Ref. 19), except that (asymptotic) identities are replaced by (asymptotic) inequalities, inequalities that should only be saturated above the upper critical dimension.

The exponent δ may be defined either by $P_n(p_c) \sim n^{-(1+1/\delta)}$ or by

$$\chi(p_c; h) \equiv \sum_n n e^{-nh} P_n(p_c) \sim h^{1/\delta - 1} \quad \text{as} \quad h \downarrow 0 \qquad (2.8)$$

Now the lattice animal representation (2.2) imples

$$P_{nl}(p_c - \varepsilon) = \psi_{nl}(\varepsilon) P_{nl}(p_c) \qquad (2.9)$$

where

$$\psi_{nl}(\varepsilon) = (1 - \varepsilon/p_c)^n [1 + \varepsilon/(1-p_c)]^l \qquad (2.10)$$

so that

$$\chi(p_c - \varepsilon) = \sum_{n,l} n \psi_{nl}(\varepsilon) P_{nl}(p_c) \qquad (2.11)$$

The identity (2.6), which is valid for $p < p_c$, suggests that when $p = p_c$, $[n/p_c - l(1-p_c)]$ is "typically" $O(n^{1/2})$ as $n \to \infty$. In this typical region of (n, l) values, $l \geqslant (1-p_c)n/p_c - O(n^{1/2})$ and

$$\psi_{nl}(\varepsilon) \geqslant (1 - \varepsilon/p_c)^n [1 + \varepsilon/(1-p_c)]^{(1-p_c)n/p_c - O(n^{1/2})} \qquad (2.12)$$

If we expand the logarithm of the RHS of (2.12) in ε for small ε and use (2.11), we find

$$\chi(p_c - c) \geqslant \sum_n n e^{-O(n\varepsilon^2 + n^{1/2}\varepsilon)} P_n(p_c) \tag{2.13}$$

Since $n^{1/2}\varepsilon \leqslant (1 + n\varepsilon^2)/2$, this last inequality and the definition (2.8) of δ together show that

$$\chi(p_c - \varepsilon) \geqslant O[\chi(p_c; \varepsilon^2)] \sim \varepsilon^{-2(1-1/\delta)}$$

which implies $\gamma \geqslant 2(1 - 1/\delta)$. Similar arguments lead to $\gamma' \geqslant 2(1 - 1/\delta)$.

The major change made below to turn the above discussion into a legitimate proof is that $[n/p - l/(1-p)]$ is only shown to be $O[(n \log n)^{1/2}]$ rather than $O(n^{1/2})$. This leads to the logarithm in the conclusion (1.10) of the theorem.

Proof of Theorem 2 (for site percolation). We give the proof of (1.10) for $\chi(p_c - \varepsilon)$; the proof for $\chi'(p_c + \varepsilon)$ is essentially the same. For some K_2 in $(0, \infty)$, whose value will be implicitly determined below, we define

$$\sum_a = \sum_{l:|n/p_c - l/(1-p_c)| \leqslant K_2(n \log n)^{1/2}/(1-p_c)}$$

$\sum_b$ to be the complementary sum, and

$$\psi(n, \varepsilon) = (1 - \varepsilon/p_c)^n [1 + \varepsilon/(1 - p_c)]^{(1-p_c)n/p_c - K_2(n \log n)^{1/2}}$$

Then the lattice animal identities (2.9)–(2.10) imply

$$P_n(p_c - \varepsilon) \geqslant \sum_a \psi_{nl}(\varepsilon) P_{nl}(p_c) \geqslant \psi(n, \varepsilon) \left[P_n(p_c) - \sum_b P_{nl}(p_c) \right] \tag{2.14}$$

It is known (Ref. 10, Lemma 5.1) (see also Ref. 4) that for any $K_1 < \infty$, K_2 can be chosen large enough so that for any given p (e.g., $p = p_c$)

$$\sum_b P_{nl}(p) = O(n^{-K_1}) \qquad \text{as} \quad n \to \infty \tag{2.15}$$

This allows us to convert (2.14) to

$$\chi(p_c - \varepsilon) \geqslant \sum_n \psi(n, \varepsilon) n P_n(p_c) - \sum_n \psi(n, \varepsilon) n^{-(K_1 - 1)} \tag{2.16}$$

It is not hard to show that $\psi(n, \varepsilon) \leqslant 1$ (even with $K_2 = 0$), so that for $K_1 > 2$,

$$\chi(p_c - \varepsilon) \geqslant \sum_n \psi(n, \varepsilon) n P_n(p_c) - O(1) \qquad \text{as} \quad \varepsilon \downarrow 0 \tag{2.17}$$

Expanding $\log[\psi(n, \varepsilon)]$ in ε leads to the bound (for small ε)

$$\psi(n, \varepsilon) \geqslant \exp[-K_3 \varepsilon^2 n - K_4 \varepsilon (n \log n)^{1/2}]$$

for some K_3 and K_4. We then insert the basic hypothesis (1.9) about $P_n(p_c)$ into (2.17) and estimate the sum on the RHS by

$$I(\varepsilon) \equiv \int_2^\infty du\, u^{-1/\delta} \exp[-K_3 \varepsilon^2 u - K_4 \varepsilon (u \log u)^{1/2}] \tag{2.18}$$

By the change of variables $v = \varepsilon^2 |\log(|\varepsilon|)| u$, one finds that

$$|\varepsilon^2 \log(|\varepsilon|)|^{1-1/\delta} I(\varepsilon) \to \int_0^\infty dv\, v^{-1/\delta} \exp[-K_4 (2v)^{1/2}] \qquad \text{as} \quad \varepsilon \downarrow 0$$

which yields the desired asymptotic lower bound on $\chi(p_c - \varepsilon)$. ▮

ACKNOWLEDGMENTS

The author thanks M. Aizenman, D. J. Barsky, J. T. Chayes, L. Chayes, and L. S. Schulman for their comments on a preliminary version of this paper. This research was supported in part by NSF grant DMS-8514834.

REFERENCES

1. M. Aizenman and D. J. Barsky, Sharpness of the phase transition in percolation models, Rutgers University preprint, (1986).
2. M. Aizenman and D. J. Barsky, in preparation.
3. M. Aizenman, J. T. Chayes, L. Chayes, J. Imbrie, and C. M. Newman, An intermediate phase with slow decay of correlations in one-dimensional $1/|x-y|^2$ percolation, Ising and Potts models, in preparation.
4. M. Aizenman, H. Kesten, and C. M. Newman, Uniqueness of the infinite cluster and continuity of connectivity functions for short and long range percolation, *Commun. Math. Phys.*, submitted.
5. M. Aizenman and C. M. Newman, Tree graph inequalities and critical behavior in percolation models, *J. Stat. Phys.* **36**:107–143 (1984).
6. M. Aizenman and C. M. Newman, Discontinuity of the percolation density in one-dimensional $1/|x-y|^2$ percolation models, *Commun. Math. Phys.*, to appear.
7. J. van den Berg and M. Keane, On the continuity of the percolation probability function, *Contemp. Math.* **26**:61–65 (1984).
8. J. T. Chayes and L. Chayes, An inequality for the infinite cluster density in Bernoulli percolation, *Phys. Rev. Lett.* **56**:1619–1622 (1986).
9. J. M. Hammersley, Percolation processes. Lower bounds for the critical probability, *Ann. Math. Stat.* **28**:790–795 (1957).
10. H. Kesten, *Percolation Theory for Mathematicians* (Birkhäuser, 1982).

11. H. Kesten, A scaling relation at criticality for 2D-percolation. in *Proceedings of the IMA Workshop on Percolation Theory and Ergodic Theory of Infinite Particle Systems*, H. Kesten, ed. (Springer-Verlag, to appear).
12. H. Kesten, Scaling relations for 2D-percolation, Institute for Mathematics and its Applications (Minneapolis) preprint (1986).
13. C. M. Newman, Shock waves and mean field bounds. Concavity and analyticity of the magnetization at low temperature, Appendix to Percolation theory: A selective survey of rigorous results, in *Proceedings of the SIAM Workshop on Multiphase Flow*, G. Papanicolaou, ed., to appear.
14. C. M. Newman, Inequalities for γ and related critical exponents in short and long range percolation, in *Proceedings of the IMA Workshop on Percolation Theory and Ergodic Theory of Infinite Particle Systems*, H. Kesten, ed. (Springer-Verlag, to appear).
15. M. P. M. den Nijs, A relation between the temperature exponents of the eight-vertex and q-state Potts models, *J. Phys. A* **12**:1857–1868 (1979).
16. B. Nienhuis, E. K. Riedel, and M. Schick, Magnetic exponents of the two-dimensional q-state Potts model, *J. Phys. A* **13**:L189–192 (1980).
17. C. M. Newman and L. S. Schulman, One-dimensional $1/|j-i|^s$ percolation models: The existence of a transition for $s \leqslant 2$, *Commun. Math. Phys.* **104**:547–571 (1986).
18. R. P. Pearson, Conjecture for the extended Potts model magnetic eigenvalue, *Phys. Rev. B* **22**:2579–2580 (1980).
19. D. Stauffer, Scaling properties of percolation clusters, in *Disordered Systems and Localization*, C. Castellani, C. Di Castro, and L. Peliti, eds. (Springer, 1981), pp. 9–25.